Linear Algebra

[KOMS119602] - 2022/2023

12.1 - Linear Transformation

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Matrix Transformation

(page 75 of Elementary LA Applications book)

Transformation

Definition

If f is a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$, then we say that f is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, or that f maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.

$$f \; : \; \mathbb{R}^n \to \mathbb{R}^m$$

When $m = n$, a transformation is often called an operator on $\mathbb{R}^n$.

Terminology:

- Domain:
- Codomain:

Transformation arise from linear systems

Given a linear system:

$$\begin{array}{ccccccccc} w_1 & = & a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n \\ w_2 & = & a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n \\ \vdots & & \vdots & & \vdots & & \ddots & & \vdots \\ w_m & = & a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n \end{array}$$

which can be written in matrix notation $\mathbf{w} = A\mathbf{x}$:

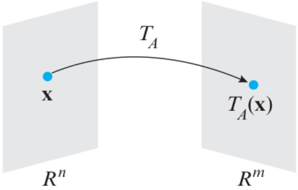
$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

This can be viewed as a transformation that maps a vector $\mathbf{x} \in \mathbb{R}^n$ into the vector $\mathbf{w} \in \mathbb{R}^m$ by multiplying $\mathbf{x}$ on the left by A.

Matrix transformation

The matrix that transform a vector $\mathbf{x} \in \mathbb{R}^n$ into the vector $\mathbf{w} \in \mathbb{R}^m$ is called a matrix transformation (or a matrix operator when $m = n$), and denoted by:

$$T : \mathbb{R}^n \to \mathbb{R}^m$$



$T_A : R^n \to R^m$

Other notations that are often used are:

- $\mathbf{w} = T_A(\mathbf{x})$, which is called multiplication by A; or
- $\mathbf{x} \xrightarrow{T_A} \mathbf{w}$, which is read as T_A maps $\mathbf{x}$ into $\mathbf{w}$.

Example 1

Given a linear system:

$$
\begin{aligned}
w_1 &= 2x_1 - 3x_2 + x_3 - 5x_4 \\
w_2 &= 4x_1 + x_2 - 2x_3 + x_4 \\
w_3 &= 5x_1 - x_2 + 4x_3
\end{aligned}
$$

can be expressed in matrix form $\mathbf{w} = A\mathbf{x}$:

$$
\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
$$

In this case, the matrix A is the matrix that transforms $\mathbf{x}$ into $\mathbf{w}$.

For example, if $\mathbf{x} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}$, then

$$
\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}
$$

Example 2: zero transformations

If 0 is the $(m \times n)$ zero matrix, then:

$$T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$$

This means that multiplication by zero maps every vector in $\mathbb{R}^n$ into the zero vector in $\mathbb{R}^m$.

T_0 is called the zero transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Example 3: identity operators

If I is the $(n \times n)$ identity matrix, then:

$$T_I(\mathbf{x}) = I\mathbf{x} = \mathbf{x}$$

so multiplication by I maps every vector in $\mathbb{R}^n$ to itself. We call T_I the identity operator on $\mathbb{R}^n$.

Theorem

For every matrix A, the matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ has the following properties for all vectors $\mathbf{u}$ and $\mathbf{v}$, and for every scalar k.

1. $T_A(\mathbf{0}) = \mathbf{0}$
2. $T_A(k\mathbf{u}) = kT_A(\mathbf{u})$ *(homogenity property)*
3. $T_A(\mathbf{u} + \mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$
4. $T_A(\mathbf{u} - \mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$ *(additivity property)*

$\sim$ **Question** $\sim$

- Are there algebraic properties of a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ that can be used to determine whether T is a matrix transformation?
- If we discover that a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation, how can we find a matrix for it?

Linear transformation

Theorem (Linearity conditions)

$T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation if and only if the following relationships hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and for every scalar k:

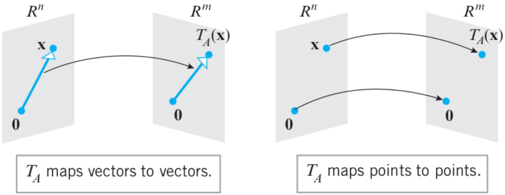
1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ *(additivity property)*
2. $T(k\mathbf{u}) = kT(\mathbf{u})$ *(homogenity property)*

A transformation that satisfies the linearity conditions is called a linear transformation

Theorem

Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation, and conversely, every matrix transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a linear transformation.

Linear transformation (*cont.*)



Theorem

If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ and $T_B : \mathbb{R}^n \to \mathbb{R}^m$ are matrix transformations, and if $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every vector $\mathbf{x} \in \mathbb{R}^n$, then $A = B$.

Proof.

$T_A(\mathbf{x}) = T_B(\mathbf{x}) \Leftrightarrow A\mathbf{x} = B\mathbf{x}, \ \forall \mathbf{x} \in \mathbb{R}^n$

Taking $\mathbf{x} = \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \in \mathbb{R}^n$ (the standard basis), yields:

$$A\mathbf{e}_j = B\mathbf{e}_j \quad \text{for } j = 1, 2, \ldots, n$$

Since $A\mathbf{e}_j$ is the j-th column of A and $B\mathbf{e}_j$ is the j-th column of B, this means that the j-th column of A and the j-th column of B are the same. Hence $A = B$.

Finding standard matrices for matrix transformation

From the previous theorem, we can conclude that:

> *There is a one-to-one correspondence between $(m \times n)$ matrices and matrix transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$.*

Matrix A is called the standard matrix for a transformation from $T_A : \mathbb{R}^n \to \mathbb{R}^m$.

If $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard basis vectors for $\mathbb{R}^n$, then the standard matrix for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by:

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2) \mid \cdots \mid T(\mathbf{e}_n)]$$

Procedure

Step 1. Find the images of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ for $\mathbb{R}^n$.

Step 2. Construct the matrix that has the images obtained in Step 1 as its successive columns. This matrix is the standard matrix for the transformation.

Example 1: finding standard matrices

Example

Find the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix}$$

Solution:

Perform Step 1:

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \text{ and } T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

So, the standard matrix is:

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)] = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}$$

Example 2: computing transformation with standard matrices

Example

Given the standard matrix for transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ as follows:

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)] = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}$$

Find $T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right)$

Solution:

$$T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 3 \end{bmatrix}$$

Example 3: finding a standard matrix

Example

Find the standard matrix for the transformation:

$$T(x_1, x_2) = (3x_1 + x_2, 2x_1 - 4x_2)$$

Solution:

Write the transformation in column vectors:

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_2 \\ 2x_1 - 4x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So, the standard matrix is: $\begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}$

Task: group discussion

1. Divide yourselves into 5 groups (so, each consists of 4-5 students.
2. Each group discusses one of the following topics (read Section 1.9, page 84 - 93)
   2.1 Network Analysis Using Linear Systems
   2.2 Design of Traffic Patterns
   2.3 A Circuit with One Closed Loop and A Circuit with Three Closed Loops
   2.4 Polynomial Interpolation by Gauss-Jordan Elimination
   2.5 Approximate Integration

   You should get additional materials if the given topic is not sufficient for your presentation (for instance, if you get the topic number 4 and 5).
3. Create a video presentation to present the result of your discussion. The duration is about 15-20 minutes, and everyone in the group must present in the same proportion.

to be continued...